Robust Covariance Matrix Estimators for Sparse Data Using Regularization and RMT

Esa Ollila[1] Frédéric Pascal[2]

[1]Aalto University, School of Electrical Engineering, Finland
esa.ollila@aalto.fi http://signal.hut.fi/~esollila/

[2]CentraleSupelec, Laboratory of Signals and Systems (L2S), France
frederic.pascal@centralesupelec.fr http://fredericpascal.blogspot.fr

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Part A (Done) Esa

Regularized M-estimators of covariance:

- M-estimation and geodesic (g-)convexity
- Regularization via g-convex penalties
- Application: regularized discriminant analysis

Part B (Here we are) Frederic

Regularized M-estimators and RMT

- Robust estimation and RMT
- Regularized M-estimators
- Application(s): DoA estimation, target detection

Outline

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- Motivations
- Results

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- M-estimators asymptotics
- Applications: ANMF and MUSIC

III. Random Matrix Theory
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- Robust RMT
- Applications to DoA estimation

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- Motivations and definitions
- Optimization and detection

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Motivations ...

Signal Processing applications

- Application reality: only observations ⇒ Unknown parameters
- Several SP applications require the covariance matrix estimation, e.g. sources localization, STAP, Polarimetric SAR classification, radar detection, MIMO, discriminant analysis, dimension reduction, PCA...
- The ultimate purpose is to characterize the system performance, not only the estimation performance ⇒ ROC curves, PD vs SNR, PFA, MSE ...

Motivations ...

Robustness: what happens when models turn to be not Gaussian anymore?

- Gaussian model $\Rightarrow$ Sample Covariance Matrix
- Outliers and other parasites
- Mismodelling
- Missing data

High dimensional problems

- Massive data
- Data size can be important...
- ... greater than the number of observations
- Link with robustness.

Some insights

∗ **Robust Estimation Theory**

- More flexible and adjustable models ⇝ *CES distributions*
- Robust family of estimators ⇝ M*-estimators*
- Regularized estimators (cf. Part A)

- M-estimators statistical properties (complex case)
- Statistical properties of M-estimators functionals (e.g. MUSIC statistic for DoA estimation, ANMF detectors...)
- Regularized Tyler Estimator (RTE) derivation and asymptotics

Some insights

$*$ (Robust) Random Matrix Theory

In many applications, the dimension of the observation m is large (HSI...)

$\Rightarrow$ The required number N of observations for estimation purposes needs to be larger: $N \gg m$ BUT this is not the case in practice! Even $N < m$ is possible

$\rightsquigarrow$ New asymptotic regime: $N \to \infty, m \to \infty$ and $\dfrac{m}{N} \to c \in [0, 1]$

- Extension of "standards" for M-estimators for particular case and for general CES distribution.
- Asymptotic distribution of the eigenvalues
- Asymptotics for the RTE
- Application to DoA estimation: robust G-MUSIC

Connections between Robust Estimation Theory and RMT

Key references of this talk

Robust Estimation Theory

- E. Ollila, D. E. Tyler, V. Koivunen, and H. V. Poor, "Complex elliptically symmetric distributions: Survey, new results and applications," *Signal Processing, IEEE Transactions on*, vol. 60, pp. 5597-5625, nov. 2012.

- F. Pascal, Y. Chitour and Y. Quek,"Generalized Robust Shrinkage Estimator and Its Application to STAP Detection Problem," *Signal Processing, IEEE Transactions on*, vol. 62, pp. 5640-5651, nov. 2014.

- A. Kammoun, R. Couillet, F. Pascal and M-S. Alouini, "Convergence and fluctuations of Regularized Tyler estimators," *Signal Processing, IEEE Transactions on*, vol. 64, pp. 1048-1060, feb. 2016.

Key references of this talk

(Robust) Random Matrix Theory

- R. Couillet, F. Pascal and J. W. Silverstein, "The Random Matrix Regime of Maronna's M-estimator with elliptically distributed samples", *Journal of Multivariate Analysis*, vol. 139, pp. 56-78, 2015.
- (R. Couillet, F. Pascal and J. W. Silverstein, "Robust Estimates of Covariance Matrices in the Large Dimensional Regime," Information Theory, IEEE Transactions on, vol. 60, pp. 7269-7278, nov 2014.)
- R. Couillet, A. Kammoun, and F. Pascal, "Second order statistics of robust estimators of scatter. Application to GLRT detection for elliptical signals," *Journal of Multivariate Analysis*, vol.143, pp. 249-274 2016.
- A. Kammoun, R. Couillet, F. Pascal, and M.-S. Alouini, "Optimal Design of the Adaptive Normalized Matched Filter Detector," *submitted*, 2016. arXiv:1501.06027

I. Introduction

II. Estimation, background and applications

- Modeling the background
- Estimating the covariance matrix
- M-estimators asymptotics
- Applications: ANMF and MUSIC

III. Random Matrix Theory

IV. Regularized M-estimators and link to RMT

V. Conclusions and perspectives

Modeling the background

Complex elliptically symmetric (CES) distributions

Let $\mathbf{z}$ be a complex circular random vector of length m. $\mathbf{z}$ follows a CES ($CE(\boldsymbol{\mu}, \boldsymbol{\Lambda}, g_{\mathbf{z}})$) if its PDF can be written

$$g_{\mathbf{z}}(\mathbf{z}) = |\boldsymbol{\Lambda}|^{-1} h_z((\mathbf{z} - \boldsymbol{\mu})^H \boldsymbol{\Lambda}^{-1} (\mathbf{z} - \boldsymbol{\mu})), \quad (1)$$

where $h_z : [0, \infty) \to [0, \infty)$ is the density generator and is such as (1) defines a PDF.

- $\boldsymbol{\mu}$ is the statistical mean
- $\boldsymbol{\Lambda}$ the scatter matrix

In general (finite second-order moment), $\mathbf{M} = \alpha \, \boldsymbol{\Lambda}$ where

- $\alpha = -2\varphi'(0)$,
- φ, the characteristic generator is defined through the characteristic function $c_{\mathbf{x}}$of $\mathbf{x}$ by $c_{\mathbf{x}}(\mathbf{t}) = \exp(i\mathbf{t}^H \boldsymbol{\mu}) \, \varphi(\mathbf{t}^H \boldsymbol{\Lambda} \mathbf{t})$

Characterizing property

- Unit complex m-sphere:

$$\boxed{\mathbb{C}S^m \triangleq \{\mathbf{z} \in \mathbb{C}^m \mid \|\mathbf{z}\| = 1\}}$$

- $\mathbf{u}$ (or $\mathbf{u}^{(m)}$) = r. v. with uniform distribution on $\mathbb{C}S^m$,

$$\boxed{\mathbf{u} \sim \mathcal{U}\left(\mathbb{C}S^m\right)}$$

Theorem (Stochastic representation theorem)

$\mathbf{z} \sim CE(\boldsymbol{\mu}, \boldsymbol{\Lambda}, h_{\mathbf{z}})$ *if and only if it admits the stochastic representation*

$$\mathbf{z} =_d \boldsymbol{\mu} + \mathcal{R}\mathbf{A}\mathbf{u}^{(k)}$$

where r. va. $\mathcal{R} \geq 0$, *called the modular variate, is independent og* $\mathbf{u}^{(k)}$ *and* $\boldsymbol{\Lambda} = \mathbf{A}\mathbf{A}^H$ *is a factorization of* $\boldsymbol{\Lambda}$, *where* $\mathbf{A} \in \mathbb{C}^{m \times k}$ *with* $k =$ *rank*(Λ).

Characterizing property

1. One-to-one relation with c.d.f $F_{\mathcal{R}}(.)$ of $\mathcal{R}$ and characteristic generator $\varphi(.)$

2. Ambiguity: both $(\mathcal{R}, \mathbf{A})$ and $(c^{-1}\mathcal{R}, c\mathbf{A}), c > 0$ are valid stochastic representations of $\mathbf{z}$ $\Rightarrow$ constraint for identifiability issues

3. Distribution of quadratic form: if rank$(\mathbf{\Lambda}) = m$, then

$$Q(\mathbf{z}) \triangleq (\mathbf{z} - \boldsymbol{\mu})^H \, \mathbf{\Lambda}^{-1} \, (\mathbf{z} - \boldsymbol{\mu}) =_d \mathcal{Q}$$

where $\mathcal{Q} \triangleq \mathcal{R}^2$ is called the 2^{nd}-order modular variate.

Estimating the covariance matrix

M-estimators

PDF not specified
$\Rightarrow$ *MLE can not be derived*
$\Rightarrow$ *M-estimators are used instead*

Let $(\mathbf{z}_1, ..., \mathbf{z}_N)$ be a N-sample $\sim CE(\mathbf{0}, \mathbf{\Lambda}, g_{\mathbf{z}})$ of length m.

The complex M-estimator of $\mathbf{\Lambda}$ is defined as the solution of

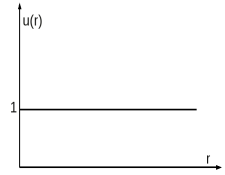
$$\mathbf{V}_N = \frac{1}{N} \sum_{n=1}^{N} u\left(\mathbf{z}_n^H \mathbf{V}_N^{-1} \mathbf{z}_n^H\right) \mathbf{z}_n \mathbf{z}_n^H, \tag{2}$$

Maronna (1976), Kent and Tyler (1991)

- Existence
- Uniqueness
- Convergence of the recursive algorithm...

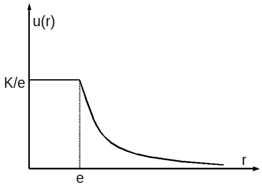
Examples of M-estimators

SCM:
$u(r) = 1$

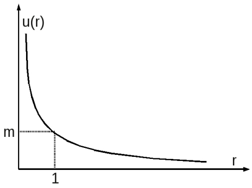


Huber's estimator (M-estimator):

$$u(r) = \begin{cases} K/e \text{ if } r <= e \\ K/r \text{ if } r > e \end{cases}$$



Tyler Estimator (Tyler, 1987; Pascal, 2008):
$u(r) = \frac{m}{r}$



Remarks:

- Huber = mix between SCM and Tyler
- FP and SCM are "not" M-estimators
- Tyler estimator is the most robust.

Tyler Estimator:

$$\mathbf{V}_N = \frac{m}{N} \sum_{n=1}^{N} \frac{\mathbf{z}_n \mathbf{z}_n^H}{\mathbf{z}_n^H \mathbf{V}_N^{-1} \mathbf{z}_n^H}$$

Context

M-estimators

Let us set

$$\mathbf{V} = E\left[u(\mathbf{z}'\mathbf{V}^{-1}\mathbf{z})\,\mathbf{z}\mathbf{z}'\right], \tag{3}$$

where $\mathbf{z} \sim CE(\mathbf{0}, \mathbf{\Lambda}, g_{\mathbf{z}})$.

- (3) admits a unique solution $\mathbf{V}$ and $\mathbf{V} = \sigma\mathbf{\Lambda} = \sigma/\alpha\,\mathbf{M}$ where σ is given by Tyler(1982),
- $\mathbf{V}_N$ is a consistent estimate of $\mathbf{V}$.

Asymptotic distribution of complex M-estimators

Theorem 1 (Asymptotic distribution of $\mathbf{V}_N$)

$$\sqrt{N}\ \text{vec}(\mathbf{V}_N - \mathbf{V}) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(\mathbf{0}, \mathbf{\Sigma}, \mathbf{\Omega}\right), \qquad (4)$$

where $\mathbb{C}\mathcal{N}$ is the complex Gaussian distribution, $\mathbf{\Sigma}$ the CM and $\mathbf{\Omega}$ the pseudo CM:

$$\mathbf{\Sigma} = \sigma_1(\mathbf{V}^T \otimes \mathbf{V}) + \sigma_2 \text{vec}(\mathbf{V})\text{vec}(\mathbf{V})^H,$$
$$\mathbf{\Omega} = \sigma_1(\mathbf{V}^T \otimes \mathbf{V})\,\mathbf{K} + \sigma_2 \text{vec}(\mathbf{V})\text{vec}(\mathbf{V})^T,$$

where $\mathbf{K}$ is the commutation matrix.

Remark: The SCM is defined as $\widehat{\mathbf{S}}_N = \frac{1}{N}\sum_{n=1}^{N} \mathbf{z}_n\mathbf{z}_n^H$ where $\mathbf{z}_n$ are complex independent circular zero-mean Gaussian with CM $\mathbf{V}$. Then,

$$\sqrt{N}\ \text{vec}(\widehat{\mathbf{S}}_N - \mathbf{V}) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(\mathbf{0}, \mathbf{\Sigma}_W, \mathbf{\Omega}_W\right)$$

$$\mathbf{\Sigma}_W = (\mathbf{V}^T \otimes \mathbf{V}) \text{ and } \mathbf{\Omega}_W = (\mathbf{V}^T \otimes \mathbf{V})\,\mathbf{K}$$

An important property of complex M-estimators

- Let $\mathbf{V}_N$ an estimate of Hermitian positive-definite matrix $\mathbf{V}$ that satisfies

$$\sqrt{N}\left(\text{vec}(\mathbf{V}_N - \mathbf{V})\right) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(\mathbf{0}, \mathbf{\Sigma}, \mathbf{\Omega}\right), \tag{5}$$

with

$$\begin{cases} \mathbf{\Sigma} = \nu_1 \mathbf{V}^T \otimes \mathbf{V} + \nu_2 \text{vec}(\mathbf{V})\text{vec}(\mathbf{V})^H, \\ \mathbf{\Omega} = \nu_1 (\mathbf{V}^T \otimes \mathbf{V})\,\mathbf{K} + \nu_2 \text{vec}(\mathbf{V})\text{vec}(\mathbf{V})^T, \end{cases}$$

where ν_1 and ν_2 are any real numbers.

e.g.

	SCM	M-estimators	Tyler
ν_1	1	σ_1	$(m+1)/m$
ν_2	0	σ_2	$-(m+1)/m^2$
...	More accurate		More robust

- Let $H(\mathbf{V})$ be a r-multivariate function on the set of Hermitian positive-definite matrices, with continuous first partial derivatives and such as $H(\mathbf{V}) = H(\alpha \mathbf{V})$ for all $\alpha > 0$, e.g. the ANMF statistic, the MUSIC statistic.

An important property of complex M-estimators

Theorem 2 (Asymptotic distribution of $H(\mathbf{V}_N)$)

$$\sqrt{N}\left(H(\mathbf{V}_N) - H(\mathbf{V})\right) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(\mathbf{0}_{r,1}, \mathbf{\Sigma}_H, \mathbf{\Omega}_H\right) \tag{6}$$

where $\mathbf{\Sigma}_H$ and $\mathbf{\Omega}_H$ are defined as

$$\mathbf{\Sigma}_H = \nu_1 H'(\mathbf{V})(\mathbf{V}^T \otimes \mathbf{V})H'(\mathbf{V})^H,$$
$$\mathbf{\Omega}_H = \nu_1 H'(\mathbf{V})(\mathbf{V}^T \otimes \mathbf{V})\,\mathbf{K}H'(\mathbf{V})^T,$$

where $H'(\mathbf{V}) = \left(\dfrac{\partial H(\mathbf{V})}{\partial \text{vec}(\mathbf{V})}\right)$.

$H(SCM)$ and $H(M$-estimators) share the same asymptotic distribution (differs from σ_1)

Application: Detection using the ANMF test

- In a m-vector $\mathbf{y}$, detecting a complex known signal $\mathbf{s} = \alpha\,\mathbf{p}$ embedded in an additive noise $\mathbf{z}$ (with covariance matrix $\mathbf{V}$), can be written as the following statistical test:

$$\begin{cases} \text{Hypothesis } H_0\text{:} & \mathbf{y} = \mathbf{z} & \mathbf{y}_n = \mathbf{z}_n & n = 1, \ldots, N \\ \text{Hypothesis } H_1\text{:} & \mathbf{y} = \mathbf{s} + \mathbf{z} & \mathbf{y}_n = \mathbf{z}_n & n = 1, \ldots, N \end{cases}$$

  where the $\mathbf{z}_n$'s are N "signal-free" independent observations (secondary data) used to estimate the noise parameters.

- Let $\mathbf{V}_N$ be an estimate of $\mathbf{V}$.

ANMF test

$$\Lambda(\mathbf{V}_N) = \frac{|\mathbf{p}^H \mathbf{V}_N^{-1} \mathbf{y}|^2}{(\mathbf{p}^H \mathbf{V}_N^{-1} \mathbf{p})(\mathbf{y}^H \mathbf{V}_N^{-1} \mathbf{y})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

One has $\Lambda(\mathbf{V}_N) = \Lambda(\alpha\,\mathbf{V}_N)$ for any $\alpha > 0$.

Probabilities of false alarm

P_{fa}-threshold relation in the Gaussian case of $\Lambda(SCM)$ (finite N)

$$P_{fa} = (1-\lambda)^{a-1} \, {}_2F_1(a, a-1; b-1; \lambda), \tag{7}$$

where $a = N - m + 2$, $b = N + 2$ and ${}_2F_1$ is the Hypergeometric function.

From theorem 2, one has both results

P_{fa}-threshold relation of $\Lambda(M\text{-est})$ for CES distributions

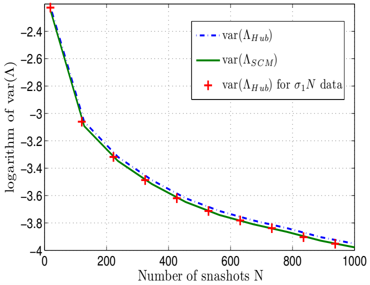
For N large and any elliptically distributed noise, the PFA is still given by (7) if we replace N by N/ν_1.

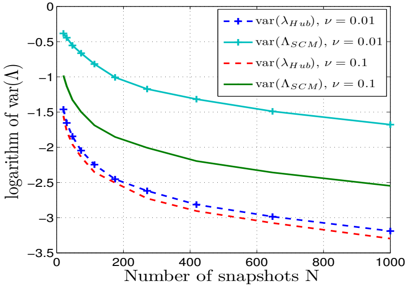
P_{fa}-threshold relation of $\Lambda(M\text{-est})$ for CES distributions

$$\sqrt{N}\,(\Lambda(\mathbf{V}_N) - \Lambda(\mathbf{V})) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(\mathbf{0}, 2\,\nu_1 \Lambda(\mathbf{V})\,(\Lambda(\mathbf{V})-1)^2\right)$$

Simulations

- Complex Huber's M-estimator.
- Figure 1: Gaussian context, here $\sigma_1 = 1.066$.
- Figure 2: K-distributed clutter (shape parameter: 0.1, and 0.01).



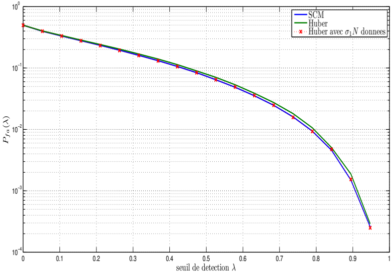


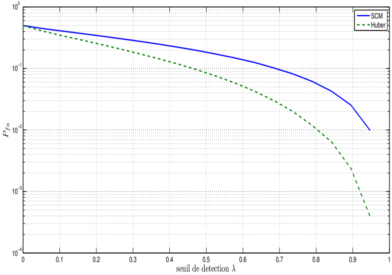
Thm validation (even for small N)

Interest of the M-estimators

Simulations: Probabilities of False Alarm

- Complex Huber's M-estimator.
- Figure 1: Gaussian context, here $\sigma_1 = 1.066$.
- Figure 2: K-distributed clutter (shape parameter: 0.1).





Validation of theorem (even for small N)

Interest of the M-estimators for False Alarm regulation

MUSIC method for DoA estimation

- K direction of arrival θ_k on m antennas
- Gaussian stationary narrowband signal with DoA $20°$ with additive noise.
- the DoA is estimated from N snapshots, using the SCM and the Huber's M-estimator.

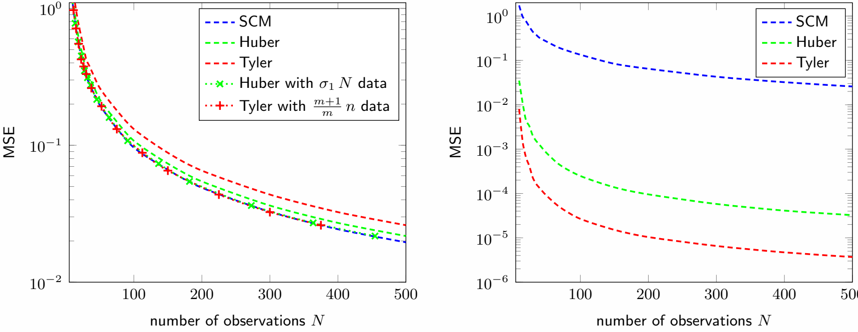
$$\mathbf{z}_t = \sum_{k=1}^{K} \sqrt{p}_k \mathbf{s}(\theta_k)\mathbf{y}_{k,t} + \sigma \mathbf{w}_t$$

$$\begin{cases} H(\mathbf{V}) & = \gamma(\theta) = \mathbf{s}(\theta)^H \mathbf{E}_W \mathbf{E}_W^H \mathbf{s}(\theta), \qquad (\mathbf{V} \text{ known}) \\ H(\mathbf{V}_N) & = \hat{\gamma}(\theta) = \sum_{i=1}^{m-K} \lambda_i \mathbf{s}(\theta)^H \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i^H \mathbf{s}(\theta) = H(\alpha\, \mathbf{V}_N), \qquad (\mathbf{V} \text{ unknown}) \end{cases}$$

where λ_i (resp. $\hat{\mathbf{e}}_i$) are the eigenvalues (resp.eigenvectors) of $\mathbf{V}_N$.
The Mean Square Error (MSE) between the estimated angle $\widehat{\theta}$ and the real angle θ is then computed (case of one source).

Simulation using the MUSIC method

- A $m = 3$ ULA with half wavelength sensors spacing is used,
- Gaussian stationary narrowband signal with DoA 20° with additive noise.
- the DoA is estimated from N snapshots, using the SCM, the Huber's M-estimator and the FP estimator.



(a) White Gaussian additive noise (b) K-distributed additive noise ($\nu = 0.1$)

Figure: MSE of $\hat{\theta}$ vs the number N of observations, with $m = 3$.

Interest of RMT: A very simple example...

Problem: Estimation of 1 DoA embedded in white Gaussian noise

$$\mathbf{z}_t = \sqrt{p}\mathbf{s}(\theta)\mathbf{y}_t + \mathbf{w}_t$$

where the $\mathbf{w}_t$'s are n independent realizations of circular white Gaussian noise, i.e. $\mathbf{w}_t \sim \mathbb{C}\mathcal{N}(0, \mathbf{I})$.

Classical approach

- $\widehat{\mathbf{S}}_N = \frac{1}{N}\sum_{t=1}^{N} \mathbf{w}_t \mathbf{w}_t^H \underset{n\to\infty}{\longrightarrow} \mathbf{I}$
- Then, MUSIC algorithm allows to estimate the DoA...

What happens when the dimension m is large?

- $\widehat{\mathbf{S}}_N = \frac{1}{N}\sum_{t=1}^{N} \mathbf{w}_t \mathbf{w}_t^H \underset{m,n\to\infty}{\not\longrightarrow} \mathbf{I}$
- Then, MUSIC algorithm IS NOT the best way to estimate the DoA...

Classical approach: $N \gg m$

e.g. STAP context, 4 sensors and 64 pulses, $m = 256$ and $N = 10^4$

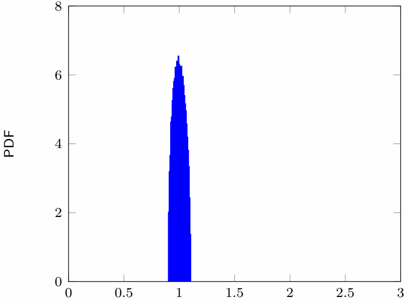


Figure: Empirical distribution for the eigenvalues of the SCM in the case of a white Gaussian noise of dimension $m = 256$ for $N = 10^4$ secondary data

What happens when the dimension m is large? (compared to N)

STAP context, 4 sensors and 64 pulses, $m = 256$ and $N = 10^3$

Marcenko-Pastur Law...

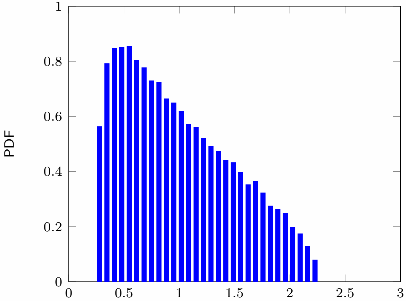


Figure: Empirical distribution for the eigenvalues of the SCM in the case of a white Gaussian noise of dimension $m = 256$ for $N = 10^3$ secondary data

What happens when the dimension m is large? (compared to N)

STAP context, 4 sensors and 64 pulses, $m = 256$ and $N = 500$

Marcenko-Pastur Law...

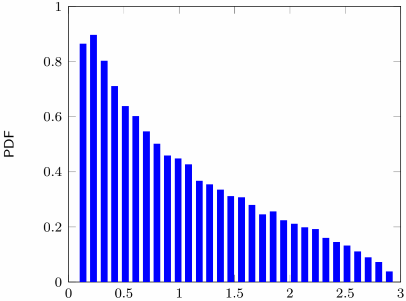


Figure: Empirical distribution for the eigenvalues of the SCM in the case of a white Gaussian noise of dimension $m = 256$ for $N = 500$ secondary data

Consequences

Bad assumptions $\Longrightarrow$ Bad performance

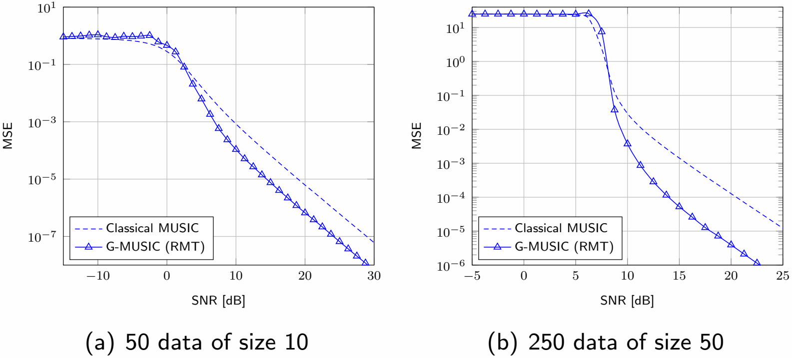


Figure: MSE on the different DoA estimators for $K = 1$ source embedded in an additive white Gaussian noise

RMT - Classical results

Assumptions:

- $N, m \to \infty$ and $\frac{m}{N} \to c \in (0,1)$ and $\widehat{\mathbf{S}}_N = \frac{1}{N}\sum_{i=1}^{N} \mathbf{z}_i \mathbf{z}_i^H$ the SCM
- $(\mathbf{z}_1, ..., \mathbf{z}_N)$ be a N-sample, i.i.d (i.e. $E[\mathbf{z}_i^{(j)}\, \mathbf{z}_k^{(l)}] = 0$) with finite fourth-order moment.

Remark: CES dist. do not respect this assumptions!

Thus one has:

1) $F^{\widehat{\mathbf{S}}_N} \Rightarrow F^{MP}$

   where $F^{\widehat{\mathbf{S}}_N}$ (resp. F^{MP}) stands for the distribution of the eigenvalues of $\widehat{\mathbf{S}}_N$ (resp. the Marcenko-Pastur distribution) and $\Rightarrow$ stands for the weak convergence.

   The MP PDF is defined by

$$\mu(x) = \begin{cases} \left(1 - \frac{1}{c}\right) \mathbf{1}_{x=0} + f(x) & \text{if } c > 1 \\ f(x) & \text{if } c \in (0,1] \end{cases}$$

with $f(x) = \frac{1}{2\pi\sigma^2} \frac{\sqrt{(c_+ - x)(x - c_-)}}{cx} \mathbf{1}_{x \in [c_-, c_+]}$ and $c_\pm = \sigma^2 \, (1 \pm \sqrt{c})^2$.

RMT - Classical results

Exploiting the MP dist for the SCM eigenvalues leads to a new MUSIC statistic:

2) $\hat{\gamma}(\theta) = \sum_{i=1}^{m} \beta_i \mathbf{s}(\theta)^H \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i^H \mathbf{s}(\theta)$ is the G-MUSIC statistic (Mestre, 2008)

where

$$\beta_i = \begin{cases} 1 + \sum_{k=m-K+1}^{m} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_i - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_i - \hat{\mu}_k} \right) & , i \leq m-K \\ -\sum_{k=1}^{m-K} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_i - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_i - \hat{\mu}_k} \right) & , i > m-K \end{cases}$$

with $\hat{\lambda}_1 \leq \ldots \leq \hat{\lambda}_m$ (resp. $\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_m$ the eigenvalues (resp. the eigenvectors) of $\widehat{\mathbf{S}}_N$ and $\hat{\mu}_1 \leq \ldots \leq \hat{\mu}_m$ the eigenvalues of $\mathrm{diag}(\hat{\boldsymbol{\lambda}}) - \frac{1}{m}\sqrt{\hat{\boldsymbol{\lambda}}}\sqrt{\hat{\boldsymbol{\lambda}}}^T$, $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_m)^T$.

Remark: Contrary to MUSIC or Robust-MUSIC, all the eigenvectors are used to compute G-MUSIC.

Robust RMT

Assumptions: (Couillet, 2014)

- $N, m \to \infty$ and $\dfrac{m}{N} \to c \in (0,1)$ and $\mathbf{V}_N$ a M-estimator (with previous assumptions)
- $(\mathbf{z}_1, ..., \mathbf{z}_N)$ be a N-sample, i.i.d (**!!!!!**) with finite fourth-order moment

Thus, it is shown that:

1) There exists a unique solution to the M-estimator fixed-point equation for all large m a.s. The recursive algorithm associated converges to this solution.

2) $\|\phi^{-1}(1)\,\mathbf{V}_N - \widehat{\mathbf{S}}_N\| \xrightarrow{a.s.} 0$ when $N, m \to \infty$ and $\dfrac{m}{N} \to c$
   where $\|.\|$ stands for the spectral norm and ϕ such that $\phi(t) = t.u(t)$.

   Remark: This result is similar to those presented in the classical asymptotic regime (m fixed and $N \to +\infty$).

Robust RMT

2) is the key result! Notably, it implies that

Classical results in RMT can be extended to the M-estimators

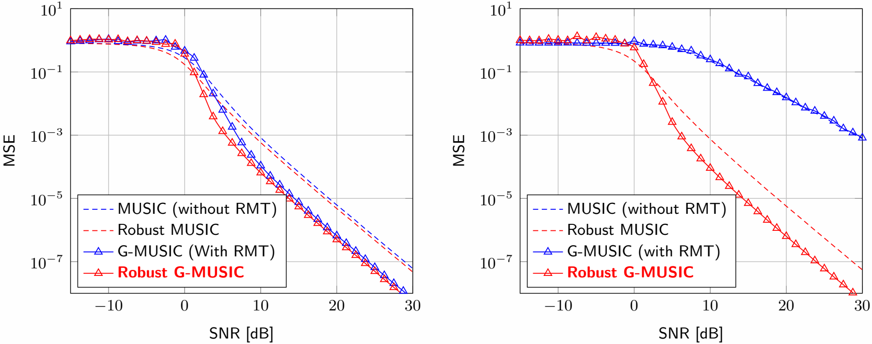
3) $\hat{\gamma}(\theta) = \sum_{i=1}^{m} \beta_i \mathbf{s}(\theta)^H \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i^H \mathbf{s}(\theta)$ is STILL the G-MUSIC statistic for the M-estimators

where

$$\beta_i = \begin{cases} 1 + \sum_{k=m-K+1}^{m} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_i - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_i - \hat{\mu}_k} \right) & , i \leq m-K \\ -\sum_{k=1}^{m-K} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_i - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_i - \hat{\mu}_k} \right) & , i > m-K \end{cases}$$

with $\hat{\lambda}_1 \leq \ldots \leq \hat{\lambda}_m$ (resp. $\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_m$ the eigenvalues (resp. the eigenvectors) of $\mathbf{V}_N$ and $\hat{\mu}_1 \leq \ldots \leq \hat{\mu}_m$ the eigenvalues of $\mathrm{diag}(\hat{\boldsymbol{\lambda}}) - \frac{1}{m}\sqrt{\hat{\boldsymbol{\lambda}}}\sqrt{\hat{\boldsymbol{\lambda}}}^T$, $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_m)^T$.

Application to DoA estimation with MUSIC for different additive clutter



(a) Homogeneous noise ($\simeq$ Gaussian), 50 data of size 10

(b) Heterogeneous clutter, 50 data of size 10

Figure: MSE performance of the various MUSIC estimators for $K = 1$ source

Resolution probability of 2 sources

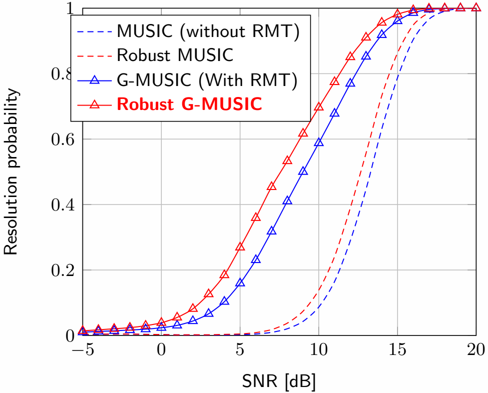


Figure: Resolution performance of the MUSIC estimators in homogeneous clutter for 50 data of size 10

Pros and Cons of these results

- Advantages
  - Original results on robust RMT
  - Now, possibility of using robust estimators in a RMT context: extension of classical RMT results such DoA estimation (done), sources power estimation, number of sources estimation (challenging problem), detection...
  - Great improvement: sources resolution, MUSIC statistic est.
- Limitations
  - Assumption of independence, i.e. not CES dist:

$$\mathbf{z}_i = \begin{pmatrix} \tau_1 \, x_i^{(1)} \\ \vdots \\ \tau_m \, x_i^{(m)} \end{pmatrix} \text{ instead of } \mathbf{z}_i = \tau_i \begin{pmatrix} x_i^{(1)} \\ \vdots \\ x_i^{(m)} \end{pmatrix}$$

    where all the quantity are independent (means $\neq$ random amplitude on the different sensors).
  - Improvement on MSE is valid for the MUSIC statistic estimate and NOT for the DoA estimate.

Robust RMT under CES distributions

- Previous results remain valid under CES distributions, i.e. where τ_i are r.va. with unknown PDF (M-estimators, (Couillet, 2015)).

  Technical condition: For each $a > b > 0$, one has
  $\lim_{t\to\infty} \frac{\limsup_N \nu_N([t,\infty))}{\phi(at)-\phi(bt)} \to 0$. where $\nu_N = \frac{1}{N}\sum_{i=1}^N \delta_{\tau_i}$ and $\phi(t) = t.u(t)$.

  Meaning: one has to control the queue of the dist. of τ_i.

- Also valid for Tyler's estimator (Zhang, 2016): $\phi(t) = m, \forall t > 0$. More tight condition but same idea for the proof.

Robust RMT under CES distributions

Results on the eigenvalues distributions of the M-estimators for CES

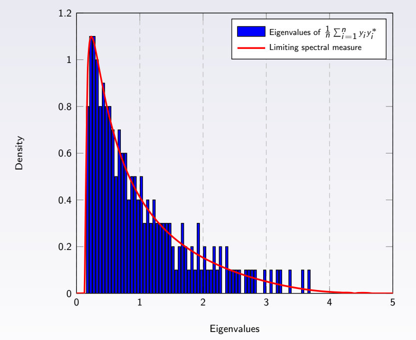
R. Couillet, F. Pascal, and J. W. Silverstein, "The Random Matrix Regime of Maronna's M-estimator with elliptically distributed samples", *JMVA*, vol. 139, 2015.

Ideas of the proofs? Break and discusssions.

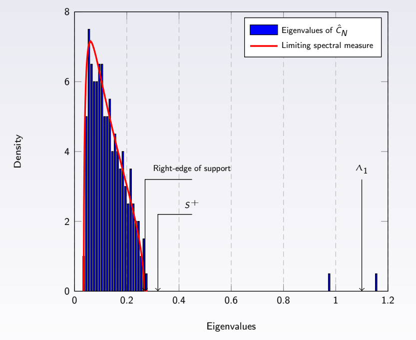
Results on the eigenvalues distributions of the Tyler's estimator for CES

T. Zhang, X. Cheng, and A. Singer,"Marchenko-Pastur Law for Tyler's and Maronna's M-estimators", arXiv preprint arXiv:1401.3424, 2016.

Robust RMT under CES distributions



(a) SCM

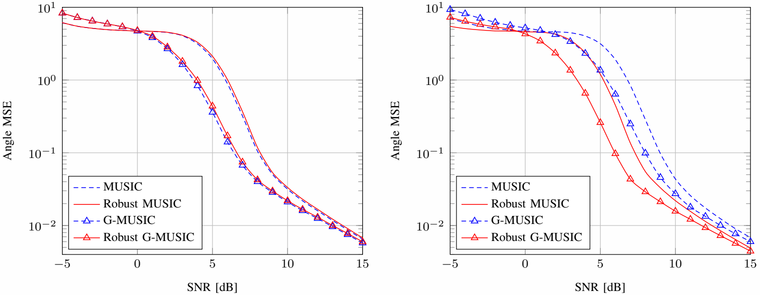


(b) Student M-estimator

Histogram of the eigenvalues of the SCM and a M-estimator against the limiting spectral measure, with 2 sources, $p_1 = p_2 = 1$, $m = 200, N = 1000$, Student-t distributions

Robust RMT under CES distributions

MSE on the DoA estimation



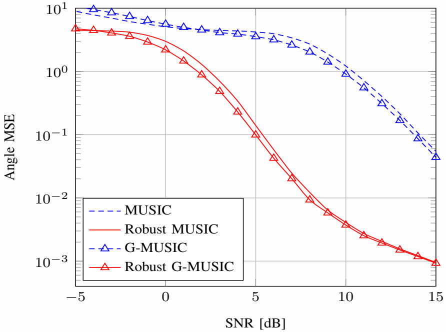
(c) Gaussian

(d) K-dist ($\nu = 1$, homogeneous)

MSE vs SNR of the DoA estimation in the case of 2 sources ($\theta_1 = 14^\circ$ and $\theta_2 = 18^\circ$), for Gaussian noise and K-distributed noise, where $N = 100$ and $m = 20$.

Robust RMT under CES distributions

MSE on the DoA estimation

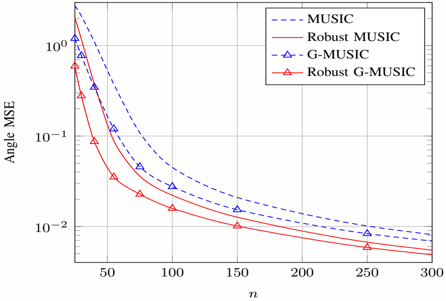


K-dist ($\nu = 0.11$, heterogeneous)

MSE vs SNR of the DoA estimation in the case of 2 sources ($\theta_1 = 14^\circ$ and $\theta_2 = 18^\circ$), for Gaussian noise and K-distributed noise, where $N = 100$ and $m = 20$.

Interest on sources resolution

Robust RMT under CES distributions



MSE vs the ration m/N of the DoA estimation in the case of 2 sources ($\theta_1 = 14^\circ$ and $\theta_2 = 18^\circ$), for homogeneous K-distributed noise, where $SNR = 10dB$ and $m = 20$.

Motivations

Some advantages

- Regularized problem (cf. part A), with norm penalties (e.g. for sparsity)
- Combined with M-estimators $\Rightarrow$ robustness to outliers
- May allow to include *a priori* informations
- Case of small number of observations or under-sampling $N < m$: matrix is not invertible $\Rightarrow$ Problem when using M-estimators or Tyler's estimator!

It is an active research on this topic:
see the works of Yuri Abramovich, Olivier Besson, Romain Couillet, Mathew McKay, Ami Wiesel...

Regularized Tyler's estimators (RTE)

Chen estimator

$$\widehat{\mathbf{\Sigma}}_C(\rho) = (1-\rho)\,\frac{m}{N}\sum_{i=1}^{N}\frac{\mathbf{z}_i\mathbf{z}_i^H}{\mathbf{z}_i^H\widehat{\mathbf{\Sigma}}_C^{-1}(\rho)\mathbf{z}_i} + \rho\mathbf{I}$$

subject to the constraint $\mathrm{Tr}(\widehat{\mathbf{\Sigma}}_C(\rho)) = m$ and for $\rho \in (0,1]$.

- Originally introduced in (Abramovich, 2007)
- Existence, uniqueness and algorithm convergence proved in (Chen, 2011)

  Y. Chen, A. Wiesel, and A. O. Hero, "Robust shrinkage estimation of high-dimensional covariance matrices," *Signal Processing, IEEE Transactions on*, vol. 59, no. 9, pp. 4097-4107, 2011.

Remark: Constraint $\mathrm{Tr}(\widehat{\mathbf{\Sigma}}_C(\rho)) = m$ has two interests:

- Allowing ρ to live in $[0,1]$
- Making the prove easier

Regularized Tyler's estimators

Pascal estimator

$$\widehat{\boldsymbol{\Sigma}}_P(\rho) = (1-\rho)\,\frac{m}{N}\sum_{i=1}^{N}\frac{\mathbf{z}_i\mathbf{z}_i^H}{\mathbf{z}_i^H\widehat{\boldsymbol{\Sigma}}_P^{-1}(\rho)\mathbf{z}_i} + \rho\mathbf{I}$$

subject to the **no** trace constraint but for $\rho \in (\bar{\rho}, 1]$, where $\bar{\rho} := \max(0, 1 - N/m)$.

- Existence, uniqueness and algorithm convergence proved in (Pascal, 2013)

  F. Pascal, Y. Chitour, and Y. Quek, "Generalized robust shrinkage estimator and its application to STAP detection problem," *Signal Processing, IEEE Transactions on,* vol. 62, pp. 5640-5651, Nov. 2014.

- $\widehat{\boldsymbol{\Sigma}}_P(\rho)$ **(naturally) verifies Tr**$(\widehat{\boldsymbol{\Sigma}}_P^{-1}(\rho)) = m$ **for all** $\rho \in (0, 1]$

Regularized Tyler's estimators

The main challenge is to find the optimal ρ!
According to the applications...**MSE, detection performances...**

One (theoretical) answer is given thanks to RMT in ...

R. Couillet and M. R. McKay, "Large Dimensional Analysis and Optimization of Robust Shrinkage Covariance Matrix Estimators," Journal of Multivariate Analysis, vol. 131, pp. 99-120, 2014.

where it is also proved that

- Both estimators have asymptotically (RMT regime) the same performance (achieved for a different value of beta)
- They asymptotically perform as a normalized version of the Ledoit-Wolf estimator (similar to previous results).

O. Ledoit and M. Wolf, "A well-conditioned estimator for large-dimensional covariance matrices," *Journal of multivariate analysis* , vol. 88, no. 2, pp. 365-411, 2004.

Regularized Tyler's estimators

Objective: Robust estimate of $\mathbf{M} = E[\mathbf{z}_i \mathbf{z}_i^H]$, for $\mathbf{z}_1, \ldots, \mathbf{z}_N \in \mathbb{C}^m$ i.i.d. with

- $\mathbf{z}_i = \sqrt{\tau_i}\,\mathbf{M}^{1/2}\mathbf{x}_i$, $\mathbf{x}_i$ has i.i.d. entries, $E[\mathbf{x}_i] = \mathbf{0}$, $E[\mathbf{x}_i \mathbf{x}_i^H] = \mathbf{I}$
- $\tau_i > 0$ random impulsions with $E[\tau_i] = 1$.
- m fixed and $N \to \infty$ (Classical asymptotics!)

OR

- $\mathbf{z}_i = \sqrt{\tau_i}\,\mathbf{M}^{1/2}\mathbf{x}_i$, $\mathbf{x}_i$ has i.i.d. entries, $E[\mathbf{x}_i] = \mathbf{0}$, $E[\mathbf{x}_i \mathbf{x}_i^H] = \mathbf{I}$
- $\tau_i > 0$ random impulsions with $E[\tau_i] = 1$.
- $c_m \triangleq \frac{m}{N} \to c$ as $m, N \to \infty$
- few data: $m \sim N$.

Find "optimal" regularized parameter!

RTE Asymptotics

Assumptions: m fixed and $N \to +\infty$

Let us set

$$\mathbf{\Sigma}_0(\rho) = m\,(1-\rho)E\left[\frac{\mathbf{z}\mathbf{z}^H}{\mathbf{z}^H\mathbf{\Sigma}_0^{-1}(\rho)\mathbf{z}}\right] + \rho\mathbf{I}$$

for $\rho \in (\bar{\rho}, 1]$, where $\bar{\rho} := \max(0, 1 - N/m)$.

Then, for any $\kappa > 0$, one has

$$\sup_{\rho\in[\kappa,1]} \left\| \widehat{\mathbf{\Sigma}}_P(\rho) - \mathbf{\Sigma}_0(\rho) \right\| \xrightarrow[m \text{ fixed }, N\to\infty]{a.s} 0$$

Remark: Of course, $\mathbf{\Sigma}_0(\rho) \neq \mathbf{M}$!!! What is $\mathbf{\Sigma}_0(\rho)$? ... it can be shown that they share the same eigenvectors space.

RTE Asymptotics

Characterization of $\Sigma_0(\rho)$

Let us first denote $\mathbf{\Sigma}_0 = \mathbf{\Sigma}_0(\rho)$.

- Multiplying by $\mathbf{M}^{-1/2}$, one obtains:

$$\mathbf{M}^{-1/2}\,\mathbf{\Sigma}_0\,\mathbf{M}^{-1/2} = m\,(1-\rho)E\left[\frac{\mathbf{x}\mathbf{x}^H}{\mathbf{x}^H\mathbf{M}^{1/2}\,\mathbf{\Sigma}_0^{-1}\,\mathbf{M}^{1/2}\mathbf{x}}\right] + \rho\mathbf{M}^{-1}$$

- Let the eigenvalue decomposition of $\mathbf{M}^{-1/2}\,\mathbf{\Sigma}_0\,\mathbf{M}^{-1/2} = \mathbf{V}\mathbf{D}\mathbf{V}^H$.
- Then, $m\,(1-\rho)E\left[\frac{\mathbf{x}\mathbf{x}^H}{\mathbf{x}^H\mathbf{D}\mathbf{x}}\right] + \rho\mathbf{V}^H\mathbf{M}^{-1}\mathbf{V} = \mathbf{D}^{-1}$

  $\Longrightarrow E\left[\frac{\mathbf{x}\mathbf{x}^H}{\mathbf{x}^H\mathbf{D}\mathbf{x}}\right] = \mathrm{diag}(\alpha_1,\ldots,\alpha_m)$ is diagonal implying $\mathbf{\Sigma}_0$ and $\mathbf{M}$ share the same eigenvector space.

Lemma If $\mathbf{D} = \mathrm{diag}(d_1,\ldots,d_m)$, then α_i are given by

$$\alpha_i = \frac{1}{2^m\,m}\,\frac{1}{\prod_{j=1}^m d_j}\,F_D^{(m)}\left(m,1,\ldots,2,1,\ldots,1,m+1,\frac{d_1-1/2}{d_1},\ldots,\frac{d_m-1/2}{d_m}\right)$$

where $F_D^{(m)}$ is the Lauricella's type D hypergeometric function.

RTE Asymptotics

Characterization of $\boldsymbol{\Sigma}_0(\rho)$

- Denote by $\alpha_i(\{d_j\}_{j=1}^m) = E\left[\frac{|x_i|^2}{\mathbf{x}^H \mathbf{D}\mathbf{x}}\right]$. Then

$$m(m-1)\alpha_i(\{d_i\}_{i=1}^m) + \frac{\rho}{\lambda_i} = \frac{1}{d_i},$$

  where λ_i are the eigenvalues of $\widehat{\boldsymbol{\Sigma}}_P(\rho)$: $\widehat{\boldsymbol{\Sigma}}_P(\rho) = \mathbf{V}\boldsymbol{\Delta}\mathbf{V}^H$ with $\boldsymbol{\Delta} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ and $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_m$.

- Start from $d_1^{(0)}, \ldots, d_m^{(0)}$ and compute iteratively

$$d_i^{(t+1)} = \frac{1}{\frac{\rho}{\lambda_i} + m(1-m)\alpha_i(\mathrm{diag}(\mathbf{d}^{(t)}))}$$

  until convergence. If $d_{1,\infty}, \ldots, d_{m,\infty}$ are the obtained values, then...

- Set $s_{i,\infty} = \lambda_i \, d_{i,\infty}$, Then,

$$\boldsymbol{\Sigma}_0 = \mathbf{V}\,\mathrm{diag}(s_{1,\infty}, \ldots, s_{m,\infty})\mathbf{V}^H.$$

RTE Asymptotics

Assumptions: m fixed and $N \to +\infty$

Similarly to M-estimators, one can establish a CLT:

Theorem 1 (Asymptotic distribution of $\widehat{\mathbf{\Sigma}}_P(\rho)$)

$$\sqrt{N}\ \text{vec}(\widehat{\mathbf{\Sigma}}_P(\rho) - \mathbf{\Sigma}_0(\rho)) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(\mathbf{0}, \mathbf{M}_1, \mathbf{M}_2\right), \qquad (8)$$

where $\mathbb{C}\mathcal{N}$ is the complex Gaussian distribution, $\mathbf{M}_1$ the CM and $\mathbf{M}_2$ the pseudo CM.

RMT Asymptotic Behavior

Theorem (Asymptotic Behavior (Couillet-McKay, 2014))

For $\varepsilon \in (0, \min\{1, c^{-1}\})$*, define* $\hat{\mathcal{R}}_\varepsilon = [\varepsilon + \max\{0, 1 - c^{-1}\}, 1]$.

Then, as $m, N \to \infty$*,* $m/N \to c \in (0, \infty)$*,*

$$\sup_{\rho \in \hat{\mathcal{R}}_\varepsilon} \left\| \widehat{\mathbf{\Sigma}}_P(\rho) - \widetilde{\mathbf{S}}_m(\rho) \right\| \xrightarrow{\text{a.s.}} 0$$

with

$$\widetilde{\mathbf{S}}_m(\rho) = \frac{1}{\underline{\gamma}(\rho)} \frac{1-\rho}{1-(1-\rho)c} \frac{1}{N} \sum_{i=1}^{N} \mathbf{M}^{\frac{1}{2}} \mathbf{x}_i \mathbf{x}_i^H \mathbf{M}^{\frac{1}{2}} + \rho \mathbf{I}$$

and $\underline{\gamma}(\rho)$ *unique positive solution to equation*

$$1 = \frac{1}{m} \operatorname{Tr}\left(\mathbf{M} \left(\rho \underline{\gamma}(\rho) \mathbf{I} + (1-\rho)\mathbf{M} \right)^{-1} \right).$$

Moreover, $\rho \mapsto \underline{\gamma}(\rho)$ *continuous on* $(0, 1]$.

Asymptotic Model Equivalence

Theorem (Model Equivalence (Couillet-McKay, 2014))

For each $\rho \in (0,1]$, *there exist unique* $\underline{\rho} \in (\max\{0, 1-c^{-1}\}, 1]$ *such that*

$$\frac{\widetilde{\mathbf{S}}_m(\rho)}{\frac{1}{\underline{\gamma}(\underline{\rho})}\frac{1-\underline{\rho}}{1-(1-\underline{\rho})c} + \underline{\rho}} = (1-\rho)\frac{1}{N}\sum_{i=1}^{N}\mathbf{M}^{\frac{1}{2}}\mathbf{x}_i\mathbf{x}_i^*\mathbf{M}^{\frac{1}{2}} + \rho\mathbf{I}.$$

Besides, $(0,1] \to (\max\{0, 1-c^{-1}\}, 1]$, $\rho \mapsto \underline{\rho}$ *is increasing and onto.*

- Estimator behaves similar to impulsion-free Ledoit-Wolf estimator
- **About uniformity:** Uniformity over ρ essential to find optimal values of ρ.
- $\widetilde{\mathbf{S}}_m$ is unobservable!

Context

Hypothesis testing: Two sets of data

- Initial pure-noise data: $\mathbf{z}_1, \ldots, \mathbf{z}_N$, $\mathbf{z}_n = \sqrt{\tau_n}\,\mathbf{M}^{1/2}\mathbf{x}_n$ as before.
- $$\begin{cases} \text{Hypothesis } H_0\text{:} & \mathbf{y} = \mathbf{z} \quad\quad \mathbf{y}_n = \mathbf{z}_n \quad n = 1, \ldots, N \\ \text{Hypothesis } H_1\text{:} & \mathbf{y} = \mathbf{s} + \mathbf{z} \quad \mathbf{y}_n = \mathbf{z}_n \quad n = 1, \ldots, N \end{cases}$$

  with $\mathbf{z} = \sqrt{\tau}\,\mathbf{M}^{1/2}\mathbf{x}$, $\mathbf{s} = \alpha\,\mathbf{p}$, $\mathbf{p} \in \mathbb{C}^m$ deterministic known, α unknown.

GLRT detection test:

$$T_m(\rho) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \Gamma$$

for some detection threshold Γ where

$$T_m(\rho) \triangleq \frac{|\mathbf{y}^H \widehat{\boldsymbol{\Sigma}}_P^{-1}(\rho)\mathbf{p}|}{\sqrt{\mathbf{y}^H \widehat{\boldsymbol{\Sigma}}_P^{-1}(\rho)\mathbf{y}}\sqrt{\mathbf{p}^H \widehat{\boldsymbol{\Sigma}}_P^{-1}(\rho)\mathbf{p}}}.$$

Context

Originally found to be $\widehat{\mathbf{\Sigma}}_P(0)$ but

- *only valid for $m < N$*
- *$\rho \geq 0$ can only bring improvements.*

Basic comments:

- For $\Gamma > 0$, as $m, N \to \infty$, $m/N \to c > 0$, under H_0,

$$T_m(\rho) \xrightarrow{\text{a.s.}} 0.$$

$\Rightarrow$ Zero false alarm, trivial result.

- Non-trivial solutions for $\Gamma = \gamma/\sqrt{m}$, $\gamma > 0$ fixed.

Objectives

Objective: For finite but large m, N, solve

$$\rho^\star = \operatorname{argmin}_\rho \left\{ P\left(\sqrt{m}T_m(\rho) > \gamma\right) \right\}.$$

Several steps:

- for each ρ, central limit theorem to evaluate

$$\lim_{\substack{m,N\to\infty \\ m/N\to c}} P\left(\sqrt{m}T_m(\rho) > \gamma\right)$$

  (very involved due to intricate structure of $\widehat{\mathbf{\Sigma}}_P$)
- find minimizing ρ
- estimate minimizing ρ

Main results

Theorem (Asymptotic detector performance (Couillet-Pascal, 2015))

As $m, N \to \infty$ *with* $m/N \to c \in (0, \infty)$,

$$\sup_{\rho \in \mathcal{R}_\kappa} \left| P\left(T_m(\rho) > \frac{\gamma}{\sqrt{m}}\right) - \exp\left(-\frac{\gamma^2}{2\sigma_m^2(\underline{\rho})}\right)\right| \to 0$$

with $\rho \mapsto \underline{\rho}$ *aforementioned mapping and*

$$\sigma_m^2(\underline{\rho}) \triangleq \frac{1}{2} \frac{\mathbf{p}^H \mathbf{M} Q_m^2(\underline{\rho}) \mathbf{p}}{\mathbf{p}^H Q_m(\underline{\rho}) \mathbf{p} \cdot \frac{1}{m} \operatorname{Tr}\left(\mathbf{M} Q_m(\underline{\rho})\right) \cdot \left(1 - c(1-\underline{\rho})^2 f(-\underline{\rho})^2 \frac{1}{N} \operatorname{Tr}\left(\mathbf{M}^2 Q\right.\right.}$$

with $Q_m(\underline{\rho}) \triangleq (\mathbf{I} + (1-\underline{\rho}) f(-\underline{\rho}) \mathbf{M})^{-1}$.

- Limiting Rayleigh distribution (weak convergence to Rayleigh $R_m(\underline{\rho})$)
- **Remark:** σ_m and $\underline{\rho}$ not function of γ

$\Rightarrow$ There exists uniformly optimal ρ.

Simulation

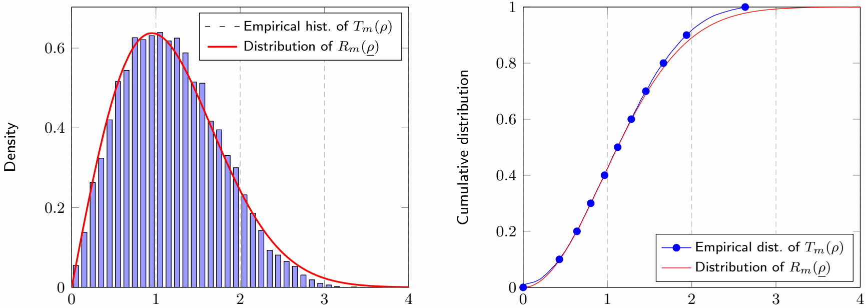


Figure: Histogram distribution function of the $\sqrt{m}T_m(\rho)$ versus $R_m(\underline{\rho})$, $m = 20$, $N = 40$ $\mathbf{p} = m^{-\frac{1}{2}}[1, \ldots, 1]^T$, $\mathbf{M}$ Toeplitz from AR of order 0.7, $\rho = 0.2$.

Simulation

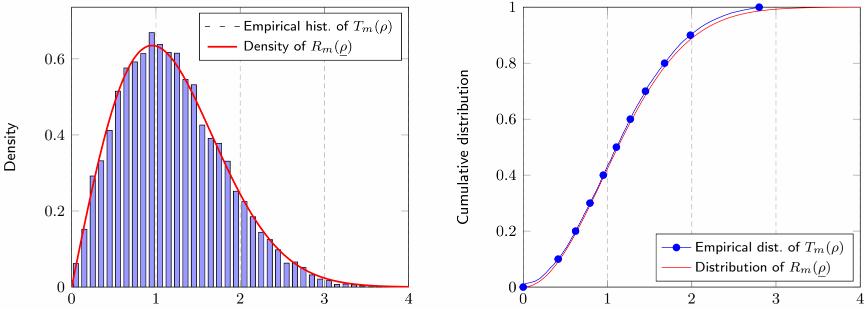


Figure: Histogram distribution function of the $\sqrt{N}T_m(\rho)$ versus $R_m(\underline{\rho})$, $m = 100$, $N = 200$ $\mathbf{p} = m^{-\frac{1}{2}}[1, \ldots, 1]^T$, $\mathbf{M}$ Toeplitz from AR of order 0.7, $\rho = 0.2$.

Empirical estimation of optimal ρ

Optimal ρ depends on unknown $\mathbf{M}$. We need:

- empirical estimate $\sigma_m(\underline{\rho})$
- minimize the estimate
- prove asymptotic optimality of estimate.

Theorem (Empirical performance estimation (Couillet-Pascal, 2015))

For $\rho \in (\max\{0, 1 - c_m^{-1}\}, 1)$, *let*

$$\hat{\sigma}_m^2(\underline{\rho}) \triangleq \frac{1}{2} \frac{1 - \rho \cdot \dfrac{\mathbf{p}^H \widehat{\mathbf{\Sigma}}_P^{-2}(\rho)\mathbf{p}}{\mathbf{p}^H \widehat{\mathbf{\Sigma}}_P^{-1}(\rho)\mathbf{p}}}{(1 - c_m + c_m \rho)\,(1 - \rho)}.$$

Also let $\hat{\sigma}_m^2(1) \triangleq \lim_{\underline{\rho} \uparrow 1} \hat{\sigma}_m^2(\underline{\rho})$.
Then

$$\sup_{\rho \in \mathcal{R}_\kappa} \left| \sigma_m^2(\underline{\rho}) - \hat{\sigma}_m^2(\underline{\rho}) \right| \xrightarrow{\text{a.s.}} 0.$$

Final result

Theorem (Optimality of empirical estimator (Couillet-Pascal, 2015))

Define

$$\underline{\rho}_m^* = \text{argmin}_{\{\rho \in \mathcal{R}'_\kappa\}} \left\{ \hat{\sigma}_m^2(\underline{\rho}) \right\}.$$

Then, for every $\gamma > 0$,

$$P\left(\sqrt{m} T_m(\underline{\rho}_m^*) > \gamma\right) - \inf_{\rho \in \mathcal{R}_\kappa} \left\{ P\left(\sqrt{m} T_m(\rho) > \gamma\right) \right\} \to 0.$$

Simulations

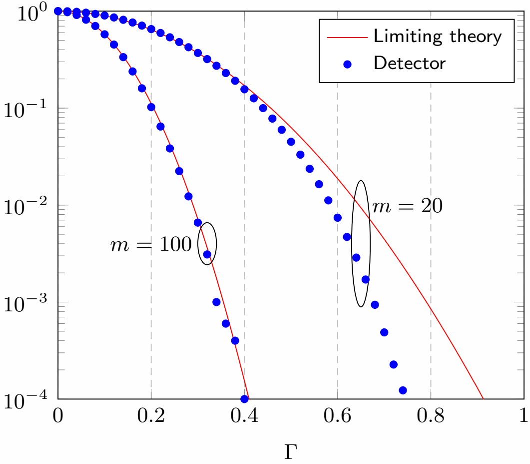


Figure: False alarm rate $P(T_m(\rho) > \Gamma)$ for $m = 20$ and $m = 100$, $\mathbf{p} = m^{-\frac{1}{2}}[1, \ldots, 1]^T$, $M_{ij} = 0.7^{|i-j|}$, $c_m = 1/2$.

Analogous results can be obtained under H_1 (more useful!).

A. Kammoun, R. Couillet, F. Pascal, and M.-S. Alouini, "Optimal Design of the Adaptive Normalized Matched Filter Detector," *Information Theory, IEEE Transactions on (submitted to)*, 2016. arXiv:1501.06027

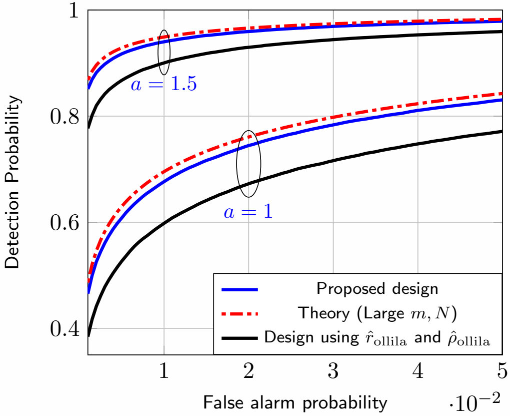


Figure: ROC curves for non-Gaussian clutters when $m = 250$ (STAP application $N_a = 10$, $N_p = 25$), $N = 250$, $f_d = 0.6$

I. Introduction

II. Estimation, background and applications

III. Random Matrix Theory

IV. Regularized M-estimators and link to RMT

V. Conclusions and perspectives

Conclusions and Perspectives

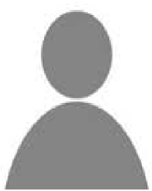
- Conclusions
  - Derivation of the complex M-estimators asymptotic distribution, the robust ANMF and the MUSIC statistic asymptotic distributions.
  - In the Gaussian case, M-estimators built with $\sigma_1 N$ data behaves as SCM built with N data (i.e. slight loss of performance in Gaussian case).
  - Better estimation in non-Gaussian cases.
  - Extension to the Robust RMT and derivation of the Robust G-MUSIC method.
  - Shrinkage M-estimators: one more degree of freedom (for Big data problems, robust methods...)

Conclusions and Perspectives

- Perspectives
  - Low Rank techniques for robust estimation
  - Robust estimation with a location parameter (non-zero-mean observation): e.g. Hyperspectral imaging
  - Second-order moment in RMT
  - Asymptotics for regularized robust estimators
  - RMT analysis for regularized robust estimators

References and thanks to...

- my co-authors:

Yacine Chitour   Abla Kammoun   Romain Couillet   Jack Silverstein

- and many inspiring people working in this field
  Yuri Abramovich, Olivier Besson, Ernesto Conte, Antonio De Maio, Alfonso Farina, Fulvio Gini, Maria Greco, Shawn Kraut, Jean-Philippe Ovarlez, Louis Scharf, Ami Wiesel . . .

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Thank you for your attention!

Questions?